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David B. Harris

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David B. Harris

Lawrence Livermore National Laboratory
P. O. Box 5504, Livermore, CA 94550

ABSTRACT

A novel 2-D rational filter design technique is presented which makes use of a reflection coefficient function (RCF) representation for the filter transfer function. The design problem is formulated in the frequency domain. A least-square error criterion is used though the usual error measure is augmented with barrier functions. These act to restrict the domain of approximation to the set of stable filters. Construction of suitable barrier functions is facilitated by the RCF characterization.

INTRODUCTION

This paper presents a frequency-domain method for the design of two-dimensional rational digital filters. Due to space limitations, its application to design in magnitude only is described. It has been used successfully for allpass filter design as well [1], and can be extended to simultaneous design in magnitude and phase.

The method presented here departs from previous efforts at recursive filter design in the manner of output mask representation. Usually these masks are specified in terms of difference equation coefficients. Here they will be characterized in terms of parameters defining reflection coefficient functions (RCFs) in a 2-D extension of the Levinson recursion. The advantage of the RCF structure is that the characterization for filter stability is made explicit in the representation. Its use affords positive control of stability in the design process.

The design technique proposed here is an optimization method employing barrier functions to restrict the search to the set of stable filters. Use of the RCF representation facilitates the construction of appropriate barrier functions.

FILTER REPRESENTATION

The filters of interest in this paper satisfy difference equations of the form:

$$\sum_{k=0}^{P} \sum_{l=-R[k]}^{R[k]} a[l,k]\, y[m-l,n-k] = \sum_{k=-N}^{N} \sum_{l=-M}^{M} b[l,k]\, x[m-l,n-k]. \quad (1)$$

The symmetry conditions

$$a[m,n] = a[-m,n]$$

$$b[m,n] = b[-m,n] = b[m,-n] = b[-m,-n]$$

are imposed to reduce the number of free design parameters in the representation. Such conditions enable the design of larger, more accurate filters, but presuppose quadrant symmetry in the desired magnitude specifications. For many applications, this condition is not restrictive.

The system transfer functions, $H(w,z)$, corresponding to (1) are ratios of polynomials of the form

$$H(w,z) = B(w,z)/A(w,z) \quad (w,z) \in L^2 \quad (2)$$

where, for example, the denominator polynomials are given by

$$A(w,z) = \sum_{n=0}^{P} \sum_{m=-R[n]}^{R[n]} a[m,n]\, w^{-m} z^{-n}. \quad (3)$$

Equations (1) and (2) are standard representations for 2-D recursive filters. We chose to use a different representation for $A(w,z)$ given by the following Levinson recursion:

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$$A(w,z) = A_p(w,z)$$

$$A_i(w,z) = A_{i-1}(w,z) + z^{-i} K_i(w) A_{i-1}(w,z^{-1})$$
$$i = 1,\ldots, P$$

$$A_0(w,z) = A(w,\infty). \qquad (4)$$

Equation (4) represents a Levinson recursion in one direction only (the index n). It is really a family of 1-D Levinson recursions parameterized by the z-transform variable w. The familiar scalar reflection coefficients of the 1-D recursion are replaced by reflection coefficient functions, $K_i(w)$. In this discussion, these are chosen to be polynominals:

$$K_i(w) = \sum_{m=-Q}^{Q} K_i[m] w^{-m} \qquad (5)$$

As remarked in the introduction, the RCF representation is attractive for its simple stability criterion. It is easy to show that any symmetric half-plane (SHP) polynomial generated by (4) is minimum phase iff

$$|K_i(w)| < 1 \quad |w| = 1 \quad i=1,\ldots,P. \qquad (6)$$

In addition, the RCF and the standard representations are completely interchangeable when rational RCF's are permitted. In this sense, the RCF representation is complete. The restriction to polynomials is not constraining in practice, however, since excellent designs have been obtained.

The output mask in the RCF representation is, likewise, generated by a Levinson recursion:

$$a[m,n] = a_P[m,n]$$

$$a_i[m,n] = a_{i-1}[m,n] + \sum_{l=-Q}^{Q} k_i[l]\, a_{i-1}[m-l, i-n]$$
$$i=1,\ldots,P$$
$$a_0[m,n] = a[m,o]. \qquad (7)$$

Since a[m,n] is constrained to be symmetric, similar constraints are implied for the reflection coefficient sequences, $\{k_i[m]\}$:

$$k_i[m] = k_i[-m] \qquad i=1,\ldots,P. \qquad (8)$$

The geometry of a typical output mask is shown in Fig. 1. At first glance, it may not appear to have a recursible implementation. However, it is possible to factor A(w,z) into a 1-D polynomial A'(w) and a nonsymmetric half-plane polynomial $A_{NSHP}(w,z)$. A'(w) is obtained by factoring $A_0(w,z)$, itself a 1-D polynomial: $A_0(w,z) = A'(w)\,A'(w^{-1})$. A_{NSHP} is generated by substituting A'(w) for $A_0(w,z)$ in the recursion of(4). This factorization leads to an implementation as a cascade of two recursible filters.

A(w,z) also has a lattice structure implementation due to its characterization with a Levinson recursion [1,2]. The lattice implementation is significantly more efficient than the corresponding difference equation implementation.

FORMULATION OF THE DESIGN PROBLEM

The objective of the design problem is to approximate a given ideal magnitude characteristic, $I(\Psi,\Omega)$ with $|F(\Psi,\Omega)|$, the magnitude of the rational filter frequency response. From equation (2), the frequency response is given by

$$F(\Psi,\Omega) = H(e^{j\Psi}, e^{j\Omega}) \quad \Psi,\Omega \in [o,\pi].$$

Due to the assumed symmetry constraints, the approximation problem is specified completely on the first quadrant of the frequency plane.

For concreteness, $I(\Psi,\Omega)$ is now taken to be piecewise constant, assuming a value of one in a desired passband and zero in a stopband. The passband and stopband are assumed to be separated by a transition region. A suitable choice of these regions for the design of a fan filter is shown in Fig. 2.

In addition, an error norm is chosen to measure the agreement between $I(\Psi,\Omega)$ and $|F(\Psi,\Omega)|$:

$$E(\underline{c}) = \iint_{PASS} (1-|F(\Psi,\Omega)|^2)^2 \, d\Psi\, d\Omega + G \iint_{STOP} |F(\Psi,\Omega)|^2 \, d\Psi\, d\Omega, \qquad (9)$$

where G is a relative weighting between the passband and the stopband, and $\underline{c}$ is the vector of parameters defining H(w,z). With this formulation, the design objective is to find a $\underline{c}^*$ minimizing E subject to the stability constraints of (6).

The barrier function method is well suited to solving this kind of constrained minimization problem. In this approach, the original problem of constrained minimization is approximated by one of unconstrained minimization. The design criterion E is augmented with a function which is negligible on most of the interior of the feasible region, but becomes sharply larger near to and finally infinite on the boundary. Thus, a descent optimization scheme starting with a point in the interior of the feasible set is forced to select among points in the interior.

In the filter design context, the constraints to be observed are (from (6) and (8)):

$$-1 < K_i(e^{j\Psi}) < 1 \quad i=1,\ldots,P \qquad \Psi \in [0,\pi].$$

For any particular Ψ, the feasible set is the P^{th} order Cartesian product of the open interval $(-1,1)$. Since the feasible set is not compact, the design problem as formulated may not have a solution vector $\underline{c}^*$. This problem is ameliorated by allowing the feasible set to include its boundary (i.e., marginally stable filters).

For any particular constraints and frequency Ψ, a suitable barrier function is:

$$B_i(\underline{c},\mu,\Psi) = \frac{1}{\mu}\,\frac{1}{1-K_i^2\,(e^{j\Psi})} \tag{10}$$

and is shown in Fig. 3. This function is infinite on the boundary of the feasible set for K_i, and may be made as small as desired in the interior by varying the accuracy parameter μ. An aggregate barrier function for all the constraints may be found by integrating the B_i over all frequencies Ψ and summing over i:

$$B(\underline{c},\mu) = \sum_{i=1}^{P} \int_0^{\pi} B_i(\underline{c},\mu,\Psi)\; d\Psi. \tag{11}$$

In the modified design problem, the criterion E is replaced by

$$E'(\underline{c}) = E(\underline{c}) + B(\underline{c},\mu)$$

and E' is minimized without constraints. The larger is μ, the more closely E' approximates E, and the more closely the solution vector $\underline{c}^*$ for the modified problem will approach that for the orginal constrained minimization problem. However, for large μ, the minimization problem may become ill-conditioned if $\underline{c}^*$ is close to the boundary. Optimization techniques used to solve the problem may then converge very slowly. In practice, experimentation is required to find a suitable choice for μ.

The numerical solution of the design problem requires that a discretization scheme be adopted. For the example shown in the next section, the continuous frequency domain $\{(\Psi,\Omega) : \Psi,\Omega \in [0,\pi]\}$ was replaced by a 33 x 33 grid of points and the integrals of (9) and (11) were approximated by sums. The self-scaling quasi-Newton method described in [3] was used to solve the minimization problem. This method requires function and gradient calculations. An analytical expression for the gradient of E' can be derived, but is not included here due to its length.

EXAMPLE

To verify the capabilities of the method proposed, a 90° fan filter with 5 RCF's is presented. The design specifications are summarized in Table 1, and the performance data in Table 2. The operation counts and storage requirements given are for a lattice implementation.

The 55 parameter fan filter was the largest attempted, requiring about an hour of CPU time to design on a PDP-11/55. No attempt was made to optimize the design algorithm for speed. For comparable designs, see [4]. The passband and stopband specifications for this filter are shown in Fig. 2. Figures 4 and 5 are a linear contour plot and a linear perspective plot (contour interval .1) of the filter magnitude, respectively.

Apart from the apparently excellent magnitude characteristics obtained, this method affords positive control of stability. This is demonstrated in Fig. 6 which displays the 5 RCF's of the fan filter. Since these lie between -1 and 1, the filter is guaranteed to be stable.

TABLE 1. Design Parameters

μ	10^5
G	1
M	4
N	4
P	5
Q	4
R[o]	3
Total parameters	55
Transition bandwidth	$\pi/8$

TABLE 2. Filter Performance

Passband Ripple	1%
Stopband Attenuation	>34 db
Multiplies/Output Point	84
Adds/Output Point	178
Rows of Storage	20

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3. Luenberger, D., Introduction to Linear and Nonlinear Programming, 1773, Addison-Wesley.

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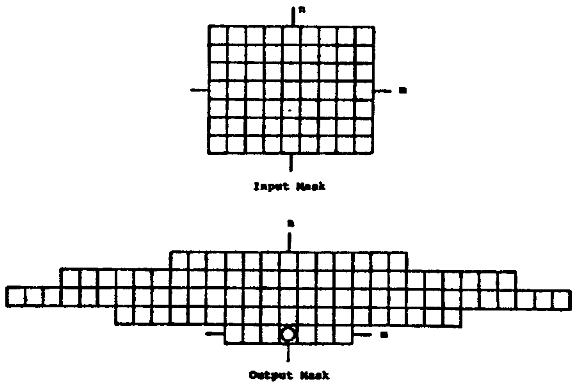

Fig. 1 Filter Masks

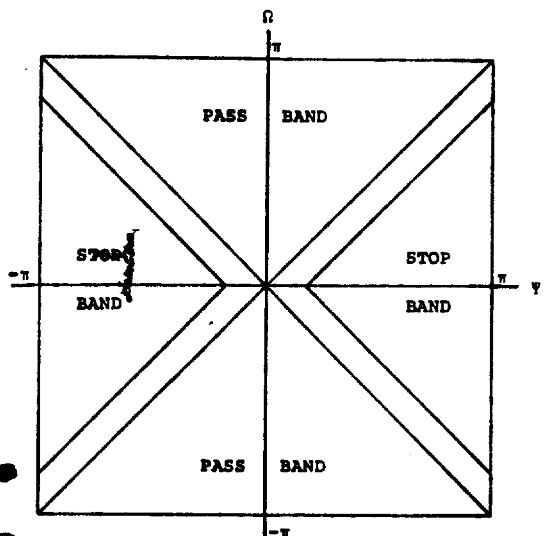

Fig. 2 Fan Filter Specifications

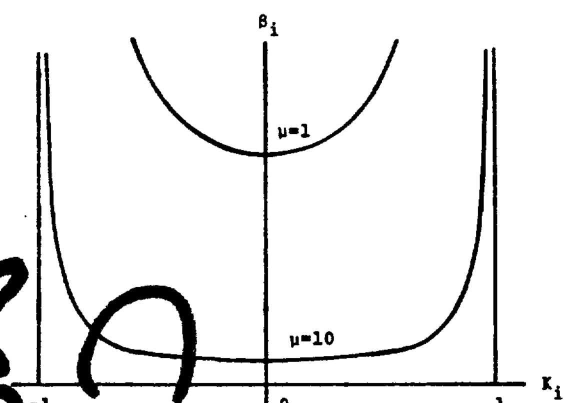

Fig. 3 Barrier Functions

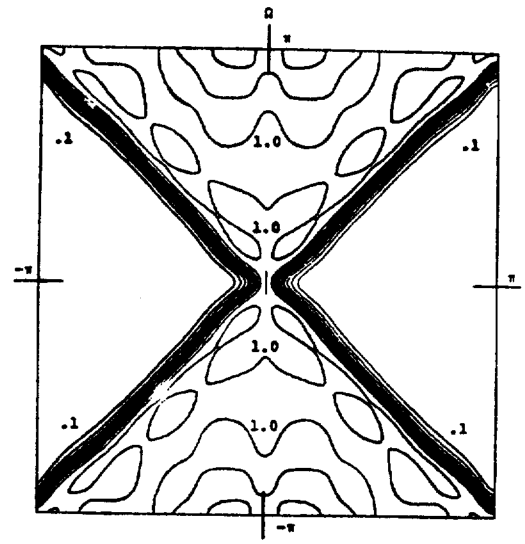

Fig. 4 Magnitude Contour Plot
Contour Interval .1

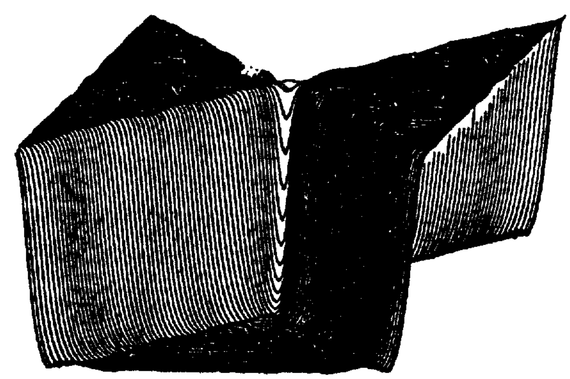

Fig. 5 Magnitude Perspective Plot

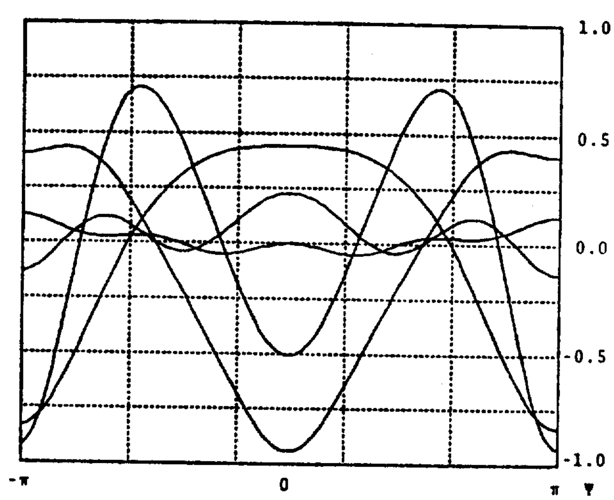

Fig. 6 Reflection Coefficient Functions